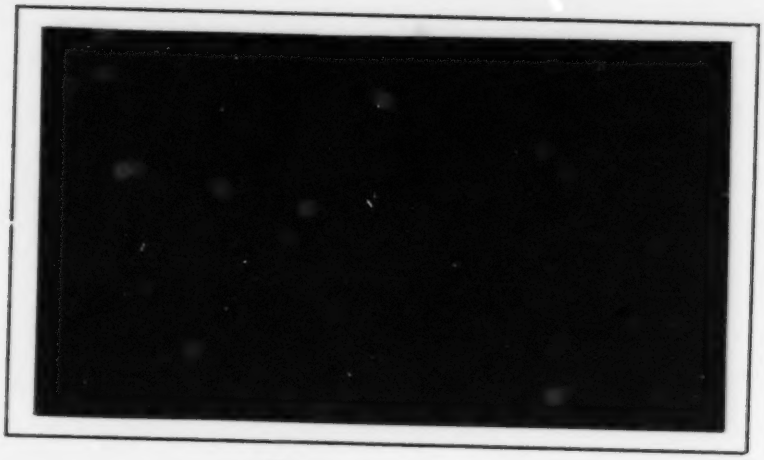

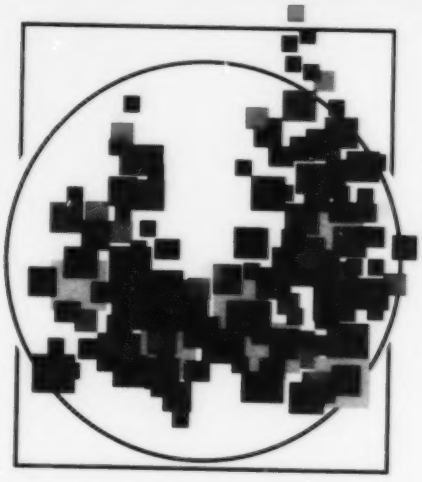

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Motion Planning in Time-varying Environments with Velocity and Acceleration Constraints: 1-D Case

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Abstract

We consider the 1-dimensional case of the motion-planning problem in time-varying environments with bounds on robot's maximum velocity and maximum acceleration. In this formulation of the problem, the aim is to find a collision-free trajectory for a point robot within a simply connected, piecewise linear space-time channel (a simple polygon in space-time) . The trajectory must satisfy the bounds on robot's maximum velocity and maximum acceleration. An $O(n^2)$ algorithm, where n is the number of vertices in the polygon, is then given for this problem. The algorithm is implemented and illustrated on some examples.

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1 Introduction

The problem of motion planning is of fundamental importance both in mobile robot navigation and in robot arm manipulation. The object of motion planning is to find a sequence of collision-free motions which is guaranteed to move the robot to its goal while avoiding obstacles in its environment. The problem has been well researched for the case of static obstacles, i.e., stationary environment. A few representative examples are [Reif 79], [Lozano-Pereź 83], [Schwartz and Sharir 85]; See [Whitesides 85] for a review.

More recently there has been a lot of interest in motion planning in the presence of moving obstacles, i.e., in time-varying environments. At the risk of belaboring the obvious, note that there is a fundamental difference between the stationary and the time-varying case: *in the former, just a path is to be planned, whereas in the latter, not only the path but the trajectory (path as a function of time) must be planned.*

A version of the motion planning problem in time-varying environments was first addressed by [Kant and Zucker 84,86].[1] They considered the problem of planning a robot's trajectory among stationary and moving (translating) polygonal obstacles. Conjecturing that the full problem is a difficult one,[2] they decomposed the problem into two stages: (i) plan a path that avoids stationary obstacles and (ii) plan the velocity of the robot along the path, so that any moving obstacles crossing the path are avoided. It is the second problem, called the *velocity planning problem* (VPP) that is of main interest here. The moving obstacles give rise to time-varying constraints on the position of the robot along the path, parameterized by s. These constraints are approximated as polygons in the path-time ($s \times t$) space. Thus, the velocity planning problem is transformed into a two dimensional path planning problem in space-time. They then used visibility graph based algorithms to find a collision-free path in ($s \times t$) space. Their algorithm incorporated the bounds on maximum velocity of the robot.

One problem with the visibility graph-based approaches is that the resulting trajectory passes too close to the obstacles. [Kant and Zucker 87] partitioned the ($s \times t$) space into obstacle-free channels – simple polygons – that are bounded by linear segments. The problem now reduces to finding a trajectory that lies completely within the channel and whose slope satisfies the maximum velocity bound. To achieve this, they gave a pruning algorithm that prunes a channel so that the regions within the channel that are not reachable or can not exited from due to maximum velocity constraint, are omitted. This process results in a pruned channel – a monotone polygon in t. A trajectory (satisfying the velocity constraint) through this monotone polygon is then easily constructed. See Figure 1. Their algorithm also incorporated splines, thus ensuring that the maximum acceleration remains finite; however, they did not take the maximum acceleration constraint into account. Thus the algorithm could result in finite but unbounded acceleration.

This problem of planning the robot's trajectory with in a simply connected region (a simple polygon in our case) has another 1-D interpretation – the point robot has to avoid two particles (one behind and the other in front of the robot) whose trajectories are given by the upper and lower boundaries (called pursuit functions) of the simple polygon. [O'Dunlaing 87] studied a version of

[1] The present author's previous publications are under the name Kamal Kant.

[2] Some complexity results on motion planning in time-varying environments have recently appeared. [Reif and Sharir 85] gave an algorithm for the case of a polygonal robot among translating polygoal obstacles; their algorithm runs in time exponential in the number of obstacles. [Reif and Canny 87] showed this problem to be NP-hard. [Fujimara and Samet 89] gives a polynomial time algorithm for the restricted case of *slowly* moving obstacles – the robot's maximum velocity being greater than the velocity of any of the obstacles.

this problem with acceleration constraints on the robot. In his elegant work, he gives polynomial time algorithms for the problem.

In this paper we consider the 1-D problem with both *velocity and acceleration constraints.* Furthermore, we allow discontinuities in the first derivative of the pursuit functions, i.e., the upper and lower pursuit functions are piecewise linear.[3] Thus, the problem considered here is, if you wish, a combination of the ones considered in [Kant and Zucker 86,87] and [O'Dunlaing 87]. And our approach to solving it draws upon the novel techniques of both these papers.

Now, we give an intuitive explanation of our algorithm. The acceleration and velocity constraints give rise to certain regions that are either inaccessible, or that can not be escaped from. In other words, these regions cannot lie on any admissible trajectories, i.e., trajectories that satisfy the velocity, acceleration and pursuit constraints.

First consider only the velocity constraints. Given the piecewise linear boundaries, as shown in [Kant and Zucker 1987], after pruning these regions, we obtain a pruned *monotone* (in t) polygon with the property that its boundaries themselves satisfy the velocity constraints. Thus, in our problem definition *we can assume that the pursuit functions already satisfy the velocity constraints.*

Now consider the acceleration constraints. Note that now certain velocities (as well as regions in $s \times t$ space) will also not be reachable. O'Dunlaing shows the structure of these regions explicitly; It suffices to say, here, that such regions are bounded by parabolic segments; however, in our case, because of the velocity constraint, additional linear segments also come into play. In nutshell, then, we first exclude (prune) any region within the free-space bounded by the pursuit functions that cannot lie on any admissible trajectory. As it shall be seen, the pruning will result in two new pursuit functions such that every point on these will lie on some admissible trajectory starting at the initial time.

Pruning can occur in two different ways as shown in Figures 2 and 3. In the first case certain regions are eliminated because of the effect of f (g) on g (f). These regions are called (after [O'Dunlaing 1987]) shadow of g (f) due to f (g), and the process of obtaining these regions is called *shading.* For instance, in Figure 2, a trajectory v with acceleration M (the acceleration bound) which touches f at only one point and passes through a concave corner of g allows the elimination of region $S1$ from the free-space since any trajectory passing through $S1$ must either cross f (or g) or violate the acceleration constraints. Similarly, region $S2$ can be eliminated because any trajectory passing through $S2$ would violate the velocity constraints.

In the second case, certain regions are eliminated because of f (g) itself (Figure 3). The process of obtaining these regions is called *filling.* For instance, in Figure 3, region S is bounded below by the trajectory δ which has acceleration $-M$ and is below G everywhere and touches g at only two points. Any trajectory passing through S must either eventually cross g transversally at some later point or violate the acceleration constraints. As already stated, since the pursuit functions already satisfy the velocity constraints (the velocity pruning having been already carried out), the relevant portion of δ will automatically satisfy the velocity constraints.

Once the pruning has been accomplished, we wish to find an admissible trajectory (if one exists) starting from initial time a and ending at a point (b, x, v) in *time-position-velocity space.* It is fairly straightforward to find such a trajectory. One illustrative way is shown in Figure 4. It involves splicing of different parabolas (acceleration $+M$ or $-M$) to the pruned admissible pursuit functions.

The paper is organized as follows: first the problem definition is given followed by two basic

[3] Our algorithm also deals with quadratic segments, however, for simplicity we will restrict our attention to linear segments only.

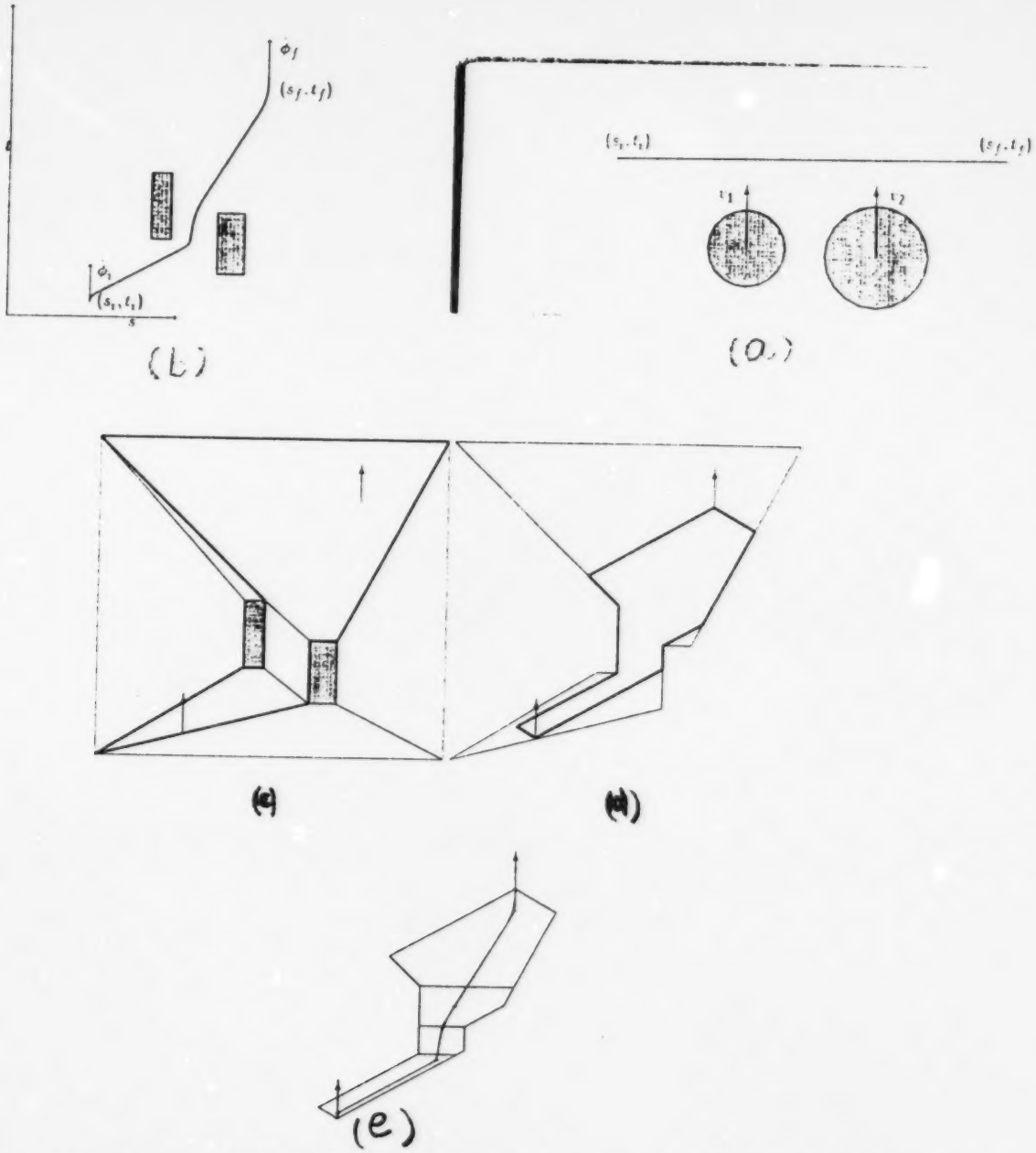

Figure 1: A motivation for the 1-D trajectory planning problem. In (a), the path of the robot is fixed – a straight line segment. The point robot is at s_i at time t_i with some initial velocity (zero in this case). The aim is to arrive at s_f at time t_f with a certain final velocity (zero in this case) while avoiding the two moving obstacles (circles). In (b), the rectangular regions in $s \times t$ space correspond to the time-varying constraints imposed on the robot's position due to obstacles' motion. Note that time is along the vertical axis. The bottom and top arrow directions represent initial and final velocities (zero in this case). In (c), a channel (a simple polygon) in space-time enclosing the initial and the final point is shown (bold lines). The problem then is to find a collision-free trajectory with in this channel – a simpler version of the original problem. In (d), the original channel and the the pruned channel that is accessible with the maximum velocity constraint are shown. The pruned channel (in bold) is a monotone (in t) polygon with the property that its boundaries themselves satisfy the maximum velocity constraint. Finally (e) shows the pruned channel only.

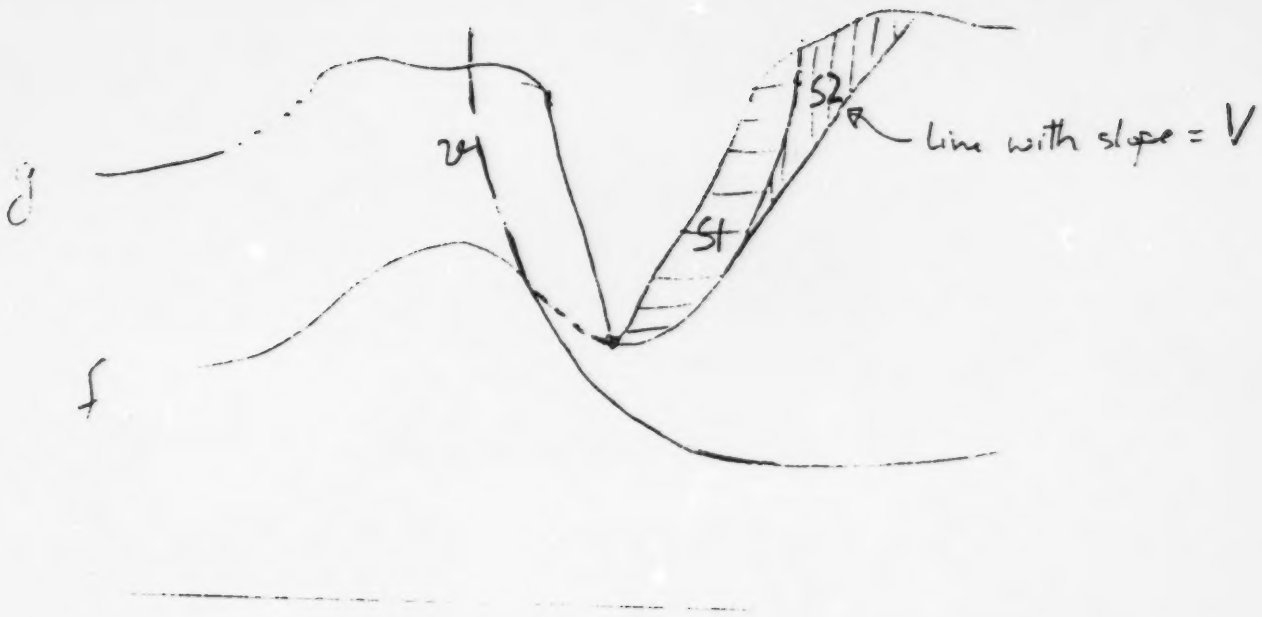

Figure 2: Shading.

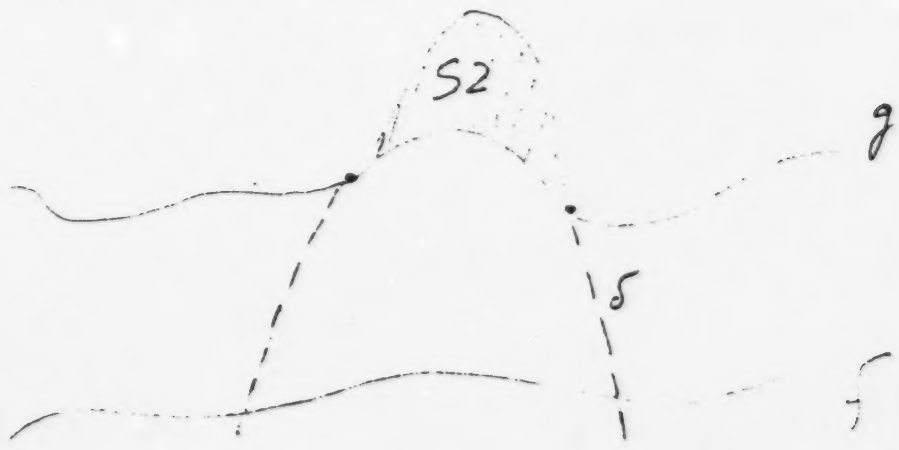

Figure 3: Filling.

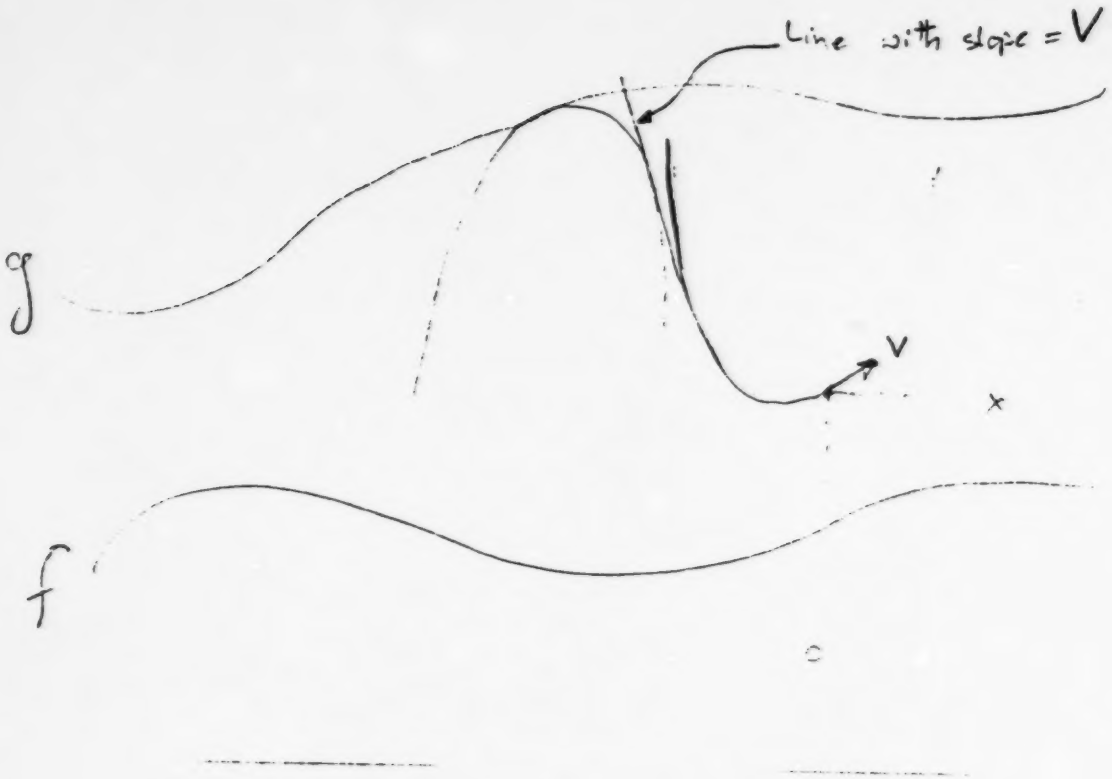

Figure 4: Trajectory reaching the point (b, x, v).

lemmas due to [O'Dunlaing 87]. Theorem 5 finds the bounds on velocities of admissible trajectories passing through a position x at time t. Lemmas 3 and 4 deal with the special case of the vertices (discontinuities in first derivative) on the pursuit functions. Then we define the two pruning operations *filling* and *shading* and prove that in fact the regions eliminated by these operations do not lie on any admissible trajectories (theorem 6). Then, we present an algorithm to make the margins reachable. Theorem 7 proves the correctness of our algorithm.

2 The Problem and Definitions

Let $V, M \in \Re^+$ be given. The two *pursuit* functions f and g defined over the interval $[a, b]$ satisfy the following conditions:

1. $f(t) \leq g(t)\ \forall t \in [a, b]$
2. Both functions are piecewise linear over the given interval.
3. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be the times at which discontinuities in g' occur in the inetrval $[a, b]$, and let $\beta_1, \beta_2, ..., \beta_m$ be the times at which discontinuities in f' occur in the inetrval $[a, b]$. Then
   $|g'(t)| \leq V\ \forall t \in [a, b]; t \neq \alpha_j, 1 \leq j \leq n$
   $|f'(t)| \leq V\ \forall t \in [a, b]; t \neq \beta_j, 1 \leq j \leq m.$

Now, we define the notion of an *admissible* trajectory. A trajectory $\bar{X}(t)$ defined over $[a, b]$ is *admissible* with respect to f and g if

1. $f(t) \leq \bar{X}(t) \leq g(t),\ \forall t \in [a, b]$ (pursuit constraints)

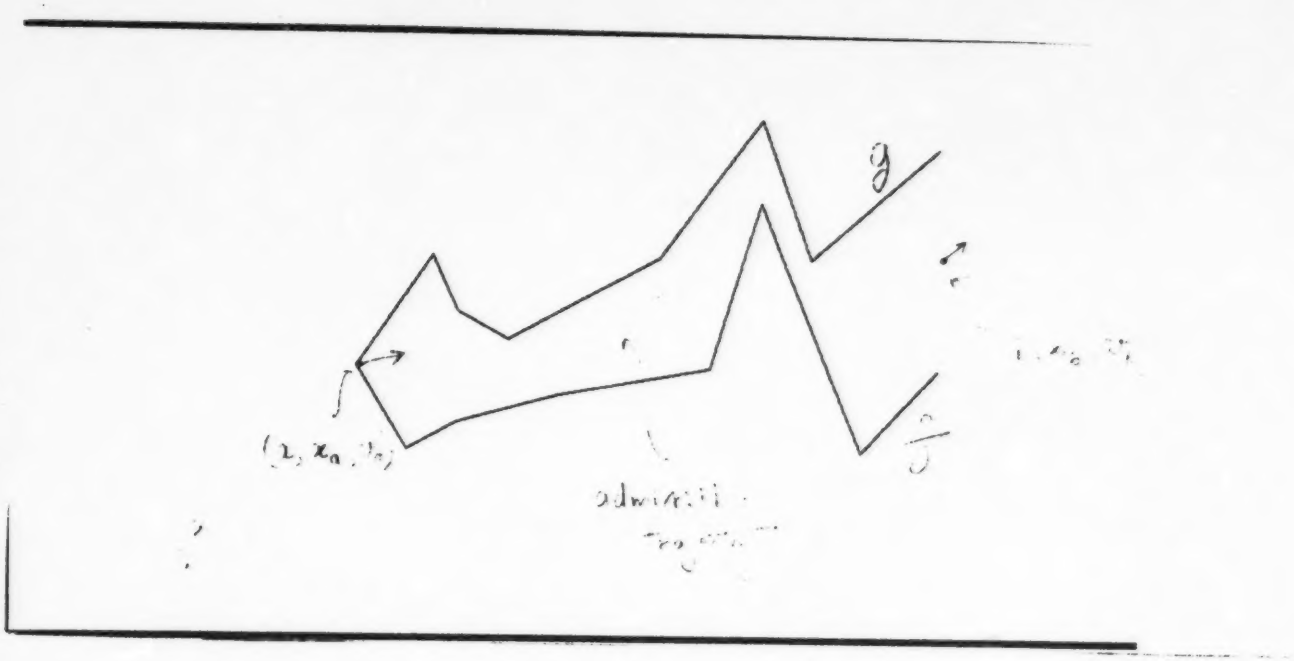

Figure 5: The problem.

2. $|\bar{X}'(t)| \leq V \ \ \forall t \in [a,b]$ (velocity constraint)

3. $|\bar{X}''(t)| \leq M \ \ \forall t \in [a,b]$ (acceleration constraint)

The *problem* then is to find an admissible trajectory $\bar{X}$ that satisfies the following initial and final conditions:

1. $\bar{X}(a) = x_a$, $\bar{X}'(a) = v_a$, (initial conditions)

2. $\bar{X}(b) = x_b$, $\bar{X}'(b) = v_b$, (final conditions)

An example is shown in Figure 5.

Now we introduce a few more definitions. We shall denote by F (G), the region bounded by $f(t)$ ($g(t)$), i.e.

$$F = \{(t,x) : a \leq t \leq b, x \leq f(t)\}$$

$$G = \{(t,x) : a \leq t \leq b, x \geq g(t)\}$$

Thus, f refers to the function and F refers to what is known as the trace of this function. Note that a discontinuity at time β_j in f' is a vertex $(\beta_j, f(\beta_j))$ of F. The same applies for G.
Next, we define the notion of reachability. A triplet (τ, x, v) is a point in time-position-velocity space depicting position x at time τ with velocity v (Figure 6). In $x \times t$ space, a triplet (τ, x, v) is represented as the point (x,t) with an arrow whose slope indicates the velocity.

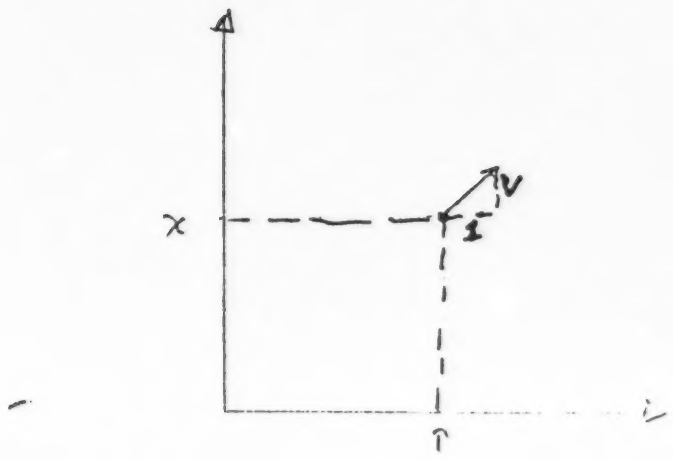

Figure 6: A point (τ, x, v).

Def: $\forall \tau \geq a$, a point (τ, x, v) is *reachable* if there exists an admissible trajectory $\bar{X}(t)$ starting at time a, satisfying $\bar{X}'(\tau) = v$ and $\bar{X}(\tau) = x$.

Def: A pursuit function f is *reachable upto time* c if $a < c \leq b$ and $\forall t \in (a, c]$

1. $\forall t = \beta_j$, $1 \leq j \leq m$, both $(t, f(t), f'_-(t))$ and $(t, f(t), f'_+(t))$ are reachable.
2. $t \neq \beta_j$, $1 \leq j \leq m$, $(t, f(t), f'(t))$ is reachable.

Def: A pair (f, g) of pursuit functions is *reachable upto time* c if $a < c \leq b$ and both f and g are *reachable* $\forall t \in (a, c]$, i.e.,

1. $\forall t = \alpha_i$, $1 \leq i \leq n$, both $(t, g(t), g'_-(t))$ and $(t, g(t), g'_+(t))$ are reachable.
2. $\forall t = \beta_j$, $1 \leq j \leq m$, both $(t, f(t), f'_-(t))$ and $(t, f(t), f'_+(t))$ are reachable.
3. $t \neq \alpha_i$, $1 \leq i \leq n$, $(t, g(t), g'(t))$ is reachable.
4. $t \neq \beta_j$, $1 \leq j \leq m$, $(t, f(t), f'(t))$ is reachable.

where, $g'_-(t)$ and $g'_+(t)$ denote the left-hand and right-hand derivatives of $g(t)$ at time t. We will show later (Lemma 4) that at a point of discontinuity t, of a reachable function $g(t)$, the whole velocity interval $[g'_-(t), g'_+(t)]$ is reachable.

Given a point $p = (\tau, x, v)$ there are unique trajectories v_p and δ_p, with accelerations M and -M respectively, passing through p:

$$\delta_p(t) = x + \frac{v^2}{2M} - \frac{1}{2}M(t - \tau - \frac{v}{M})^2$$

$$v_p(t) = x - \frac{v^2}{2M} + \frac{1}{2}M(t - \tau + \frac{v}{M})^2$$

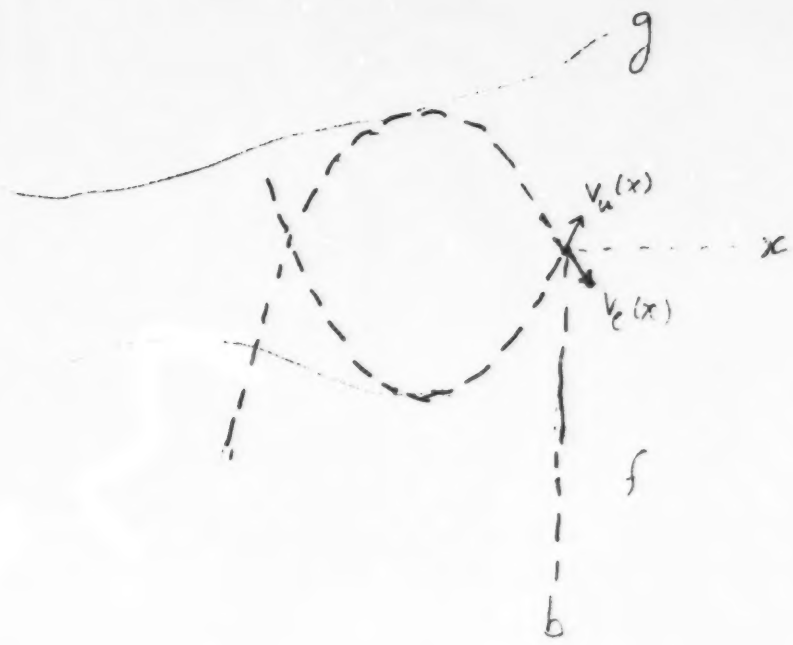

Figure 7: $\nu_u(x)$ and $\nu_l(x)$.

The next two Lemmas are due to [O'Dunlaing 87] and show some properties of $\delta_p(t)$ and $v_p(t)$. These lemmas will help us prove some necessary conditions on the reachability of the pursuit functions, and on the velocities of admissible trajectories.

Lemma 1: Let τ, x, x', v, v' be fixed, where $x \leq x'$ and $v \leq v'$ but they are not both equal. Let p, q, r, s denote $(\tau, x, v), (\tau, x, v'), (\tau, x', v), (\tau, x', v')$ respectively. Then for all points $\phi < \tau$ and $\psi > \tau$, $v_p(\psi) < v_s(\psi)$, $\delta_p(\psi) < \delta_s(\psi)$, $v_q(\phi) < v_r(\phi)$ and $\delta_q(\phi) < \delta_r(\phi)$.

Lemma 2: Suppose that f and g are reachable upto time b. If v_p meets F at some point σ earlier than τ then it cannot cross G transversally in $[\sigma, \tau]$, and if δ_p meets G at some point σ earlier than τ then it cannot cross F in $[\sigma, \tau]$.

Note that Lemma 2 holds even when allowing for discontinuities in the first derivative of f and g.

3 Necessary Conditions on Reachability and Admissibility

Def: A vertex is *convex iff* the interior angle at the vertex is less than π and is *concave* otherwise. Note that if G has a convex vertex at time t, then $g'_+(t) < g'_-(t)$, and if F has a convex vertex at time t, then $f'_+(t) > f'_-(t)$.

Def: Given time b and $f(b) \leq x \leq g(b)$, let

$$\nu_u(x) = Sup\{v : v_{(b,x,v)} \textit{ does not intersect the interior of } F \textit{ on } [a, b)\}$$

$$\nu_l(x) = Inf\{v : \delta_{(b,x,v)} \textit{ does not intersect the interior of } G \textit{ on } [a, b)\}$$

Lemma 3': Let (f,g) be a reachable pair upto time b. Then, F and G contain no convex vertices.

Proof: Assume G has a convex corner at time t. Hence, $g'_+(t) < g'_-(t)$. Therefore, for some $\sigma < t$, $v_{(t,g(t),g'_+(t))}(\sigma) > g(\sigma)$ and hence $(t, g(t), g'_+(t))$ is not reachable; a contradiction. ♠

Lemma 3: Let (f,g) be a reachable pair upto time b. Then,

$$\nu_l(g(\alpha_i)) = g'_-(\alpha_i),\ \nu_u(g(\alpha_i)) \geq g'_+(\alpha_i),\ 1 \leq i \leq n$$

$$\nu_l(f(\beta_j)) \leq f'_+(\beta_j),\ \nu_u(f(\beta_j)) = f'_-(\beta_j),\ 1 \leq j \leq m$$

Proof: Note that by Lemma 3', F and G have only concave vertices.

Case 1: $(\nu_l(f(\beta_j)) \leq f'_+(\beta_j))$. Let $v = \nu_l(f(\beta_j))$ for some j. Assume $v > f'_+(\beta_j)$. Since $\delta_{(\beta_j,f(\beta_j),v)}(\tau) < \delta_{(\beta_j,f(\beta_j),f'_+(\beta_j))}(\tau)$, $\forall \tau \in [a,\beta_j)$ and $\delta_{(\beta_j,f(\beta_j,v)}$ touches G at only one point prior to β_j, therefore $\delta_{(\beta_j,f(\beta_j),f'_+(\beta_j))}$ must intersect G transversally at some time prior to β_j. $(\beta_j, f(\beta_j), f'_+(\beta_j))$ is not reachable, a contradiction, hence $v \leq \nu_l(f(\beta_j))$.

Case 2: $(\nu_u(f(\beta_j)) = f'_-(\beta_j))$. Let $v = \nu_u(f(\beta_j))$. Then $v \leq f'_-(\beta_j)$ otherwise $v_{(\beta_j,f(\beta_j),v)}$ will intersect the interior of F . Now assume $v < f'_-(\beta_j)$. Clearly, $(\beta_j, f(\beta_j), f'_-(\beta_j))$ is reachable. By definition, $v_{(\beta_j,f(\beta_j),f'_-(\beta_j))}$ touches F at some time $\sigma' < \beta_j$. By Lemma 1, $v_{(\beta_j,f(\beta_j),f'_-(\beta_j))}(\tau) < v_{(\beta_j,f(\beta_j),\nu_u(f(\beta_j)))}(\tau)$, $\forall \tau \in [a,\beta_j)$. Therefore, $v_{(\beta_j,f(\beta_j),f'_-(\beta_j))}$ crosses F transversally and hence $(\beta_j, f(\beta_j), f'_-(\beta_j))$ is not reachable. By contradiction, $v = f'_-(\beta_j)$.
The two other cases resulting by considering g, follow by symmetrical arguments. ♠

Lemma 4: Assume f and g are a pair of pursuit functions, Assume $\forall t \in [a,b]$, $t \neq \alpha_j$, $(t, g(t), g'(t))$ is reachable, and $\forall t \in [a,b], t \neq \beta_j$, $(t, f(t), f'(t))$ is reachable. Then $\forall i, 1 \leq i \leq n$, if $v \in [\nu_l(g(\alpha_i)), \nu_u(g(\alpha_i))] \cap [-V,V]$, then $(\alpha_i, g(\alpha_i), v)$ is reachable, and $\forall j, 1 \leq j \leq m$, if $v \in [\nu_l(f(\beta_j)), \nu_u(f(\beta_j))] \cap [-V,V]$, then $(\beta_j, g(\beta_j), v)$ is reachable.

Proof: (by induction) Let $\gamma_1, \gamma_2, ..., \gamma_q$ denote the ordered sequence obtained by merging the $\alpha'_i s$ and $\beta'_j s$. We prove the lemma by induction on γ_q.
(induction step): Given γ_q assume Lemma is true $\forall i < q$. Assume $v \in [-V,V]$. Then
Case 1: ($\gamma_q = \beta_j$ for some $j \leq m$).
(i) if $v = \nu_l(f(\beta_j))$, then by the definition of $\nu_l(x)$, $\delta_{(\beta_j,f(\beta_j),v)}$ touches g at some time $\sigma < \beta_j$. If it meets g tangentially at σ , then by reachability of $(\sigma, g(\sigma), g'(\sigma))$ via a trajectory Θ, we construct a spliced trajectory Θ and δ that passes through $(\beta_j, f(\beta_j), v)$. Otherwise, if $\sigma = \alpha_i, \alpha_i < \beta_j$, then by the induction hypothesis, we know that $(\alpha_i, g(\alpha_i), v')$ $(v' = \delta'_{(\beta_j,f(\beta_j),v)}(\sigma))$ is reachable via some trajectory Θ. Note that by Lemma 3, we are guaranteed that $v' \in [\nu_l(g(\alpha_i)), \nu_u(g(\alpha_i))]$. Again using a splicing construction with Θ and δ we prove reachability.
(ii) the case $v = \nu_u(f(\beta_j))$, is symmetrical to i).
(iii) if $v \in [\nu_l(f(\beta_j)), \nu_u(f(\beta_j))]$, then (see Figure 9) consider the trajectory $\delta_{(\beta_j,f(\beta_j),v)}$ spliced to the line L at time σ where $L'(\sigma) = V$. Note that $\delta_{(\beta_j,f(\beta_j),v)}(\tau)$ cannot touch the interior of G, $\forall \tau \leq \beta_j$, since $v \geq \nu_l(f(\beta_j))$ (Lemma 1). Also, $L(\tau)$ does not touch the interior of G , $\forall \tau \leq \sigma$, since otherwise g must have $g' > V$ at some time before σ. Notice that if $\delta_{(\beta_j,f(\beta_j),v)}(\sigma) \leq f(\sigma)$, then we do not use the line-segment L. Consider the family of curves $v_{\rho(\tau)}$, where

$$\rho(\tau) = \begin{cases} (\tau, \delta_{(\beta_j,f(\beta_j),v)}(\tau), \delta'_{(\beta_j,f(\beta_j),v)}(\tau)), & \text{if } \sigma \leq \tau \leq \beta_j \\ (\tau, L(\tau), V), & \text{if } \tau < \sigma. \end{cases}$$

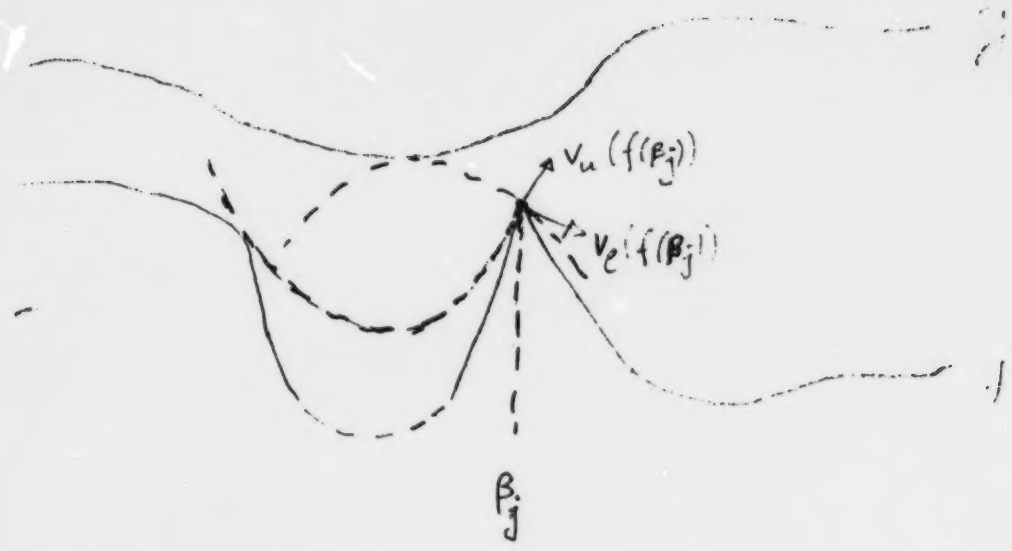

Figure 8: Admissible velocities at the vertices.

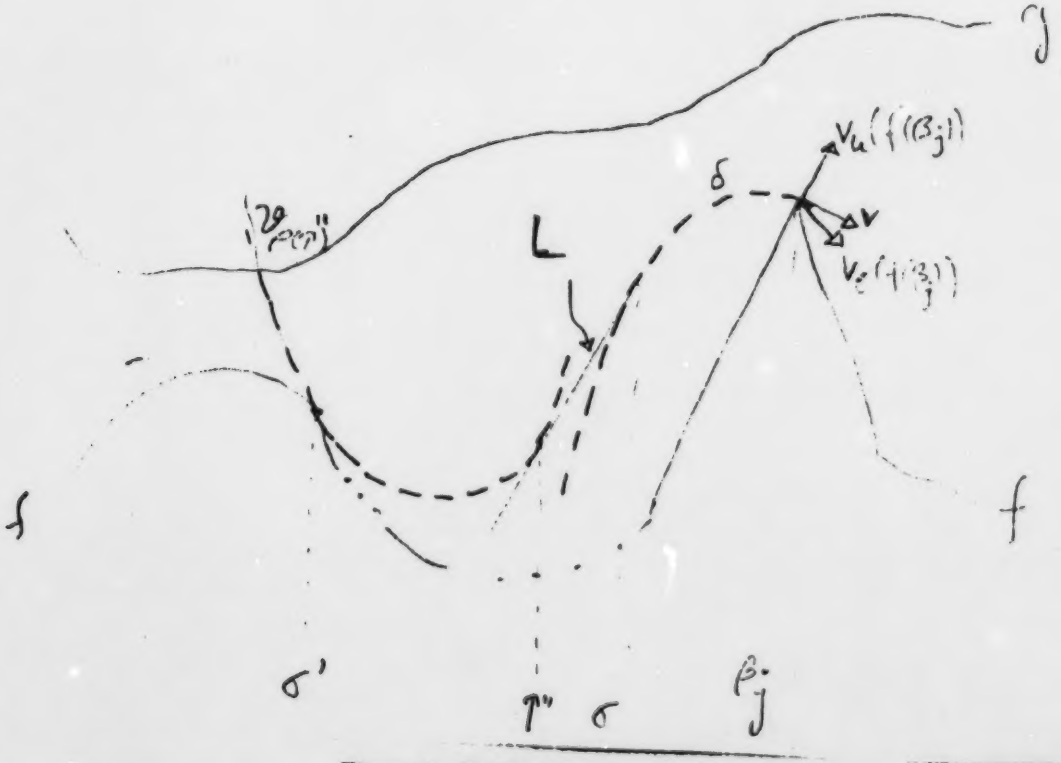

Figure 9: *Proof of Lemma 4*, $\nu_l(f(\beta_j)) < v < \nu_u(f(\beta_j))$

i.e. $\rho(\tau)$ ranges over δ and L. First note that if the trajectory obtained by splicing L to δ does not intersect the interior of F, then we are done. Otherwise, when $\tau = \beta_j$, $v_{\rho(\tau)}$ does not intersect the interior of F, and there is some time $\tau' < \tau$ where $v_{\rho(\tau')}$ intersects the interior of F. Hence, there is some intermediate time τ'' where $v_{\rho(\tau'')}$ touches F at some earlier time σ'. Note that $\forall t \in [\sigma', \tau'']$, $|v'_{\rho(\tau'')}(t)| \leq V$. Also, note that $\forall t \in [\sigma', \tau'']$, $v_{\rho(\tau'')}(t)$ does not touch the interior of G (follows by Lemma 2).

If $v_{\rho(\tau'')}$ touches F tangentially at σ', then by the reachability of $(\sigma', f(\sigma'), f'(\sigma'))$ via a trajectory Θ, Θ can be spliced to $v_{\rho(\tau'')}$, L and δ , to obtain a trajectory reaching $(\beta_j, f(\beta_j), v)$.

If $v_{\rho(\tau'')}$ touches F at β_s, $s < j$, letting $v' = v'_{\rho(\tau'')}(\beta_s)$, by induction hypothesis $(\beta_s, f(\beta_s), v')$ is reachable via some trajectory Θ. Then, again using the same splicing construction as above, we make $(\beta_j, f(\beta_j), v)$ reachable.

Case 2: ($\gamma_q = \alpha_i$, for some $i \leq n$). This case is symmetrical to case (i).

Basis: The basis case ($\gamma_q = \alpha_1$ or $\gamma_q = \beta_1$) is proved using exactly the same splicing construction as in the induction step, and hence obtain an admissible trajectory satisfying the initial conditions. ♠

Theorem 5: Assume f and g are pursuit functions reachable upto time b. Then $\forall x \in [f(b), g(b)]$ and $\forall v$, (b, x, v) is reachable iff $v \in [\nu_l(x), \nu_u(x)] \cap [-V, V]$.

Proof: ($\Rightarrow$) Assume (b, x, v) is reachable via a trajectory Θ. Since Θ intersects neither the interior of G nor the interior of F, then by Lemma 1, $v_{(b,x,v)}$ does not touch the interior of F, and $\delta_{(b,x,v)}$ does not touch the interior of G. Hence $v \in [\nu_l(x), \nu_u(x)]$ by the definition of $\nu_l(x)$ and $\nu_u(x)$. Furthermore, since Θ is admissible, it implies $v \in [-V, V]$.
($\Leftarrow$): (see Figure 10) Using the splicing construction in previous Lemma and the Lemma before, we arrive at 2 cases (ignoring $v = \nu_l(x)$ or $v = \nu_u(x)$):
Case 1: $v_{\rho(\tau'')}$ touches F tangentially at some time $\sigma < b$, for some τ''. Using the reachability of $(\sigma, f(\sigma), f'(\sigma))$ via a trajectory Θ, we construct a trajectory reaching (b, x, v).
Case 2: $v_{\rho(\tau'')}$ touches F at a point β_j with velocity v'. In this case, we invoke Lemma 4 to obtain a trajectory Θ reaching $(\beta_j, f(\beta_j), v')$, and use the splicing construction to make (b, x, v) reachable. ♠

4 Constructing the Reachable Margins

Having proved the necessary conditions on reachability and accessibility, we now can define the two basic operations "shading" and "filling" that transform a given pair of pursuit functions into a reachable pair.

Def (Filling): Let (f, g) be a pair of pursuit functions; Let

$$\bar{f}(t) = Inf\{x : \exists v(v_{(\tau,x,v)} \textit{ does not meet the interior of } F \textit{ on } [a, b)\}$$

$$\bar{g}(t) = Sup\{x : \exists v(\delta_{(\tau,x,v)} \textit{ does not meet the interior of } G \textit{ on } [a, b)\}$$

then, *filling* of f and g means to replace them with $\bar{f}$ and $\bar{g}$ repectively. Note that if $|f'| \leq V$, then $|\bar{f}'| \leq V$. Similarly, for $\bar{g}$, $|g'| \leq V \Rightarrow |\bar{g}'| \leq V$.

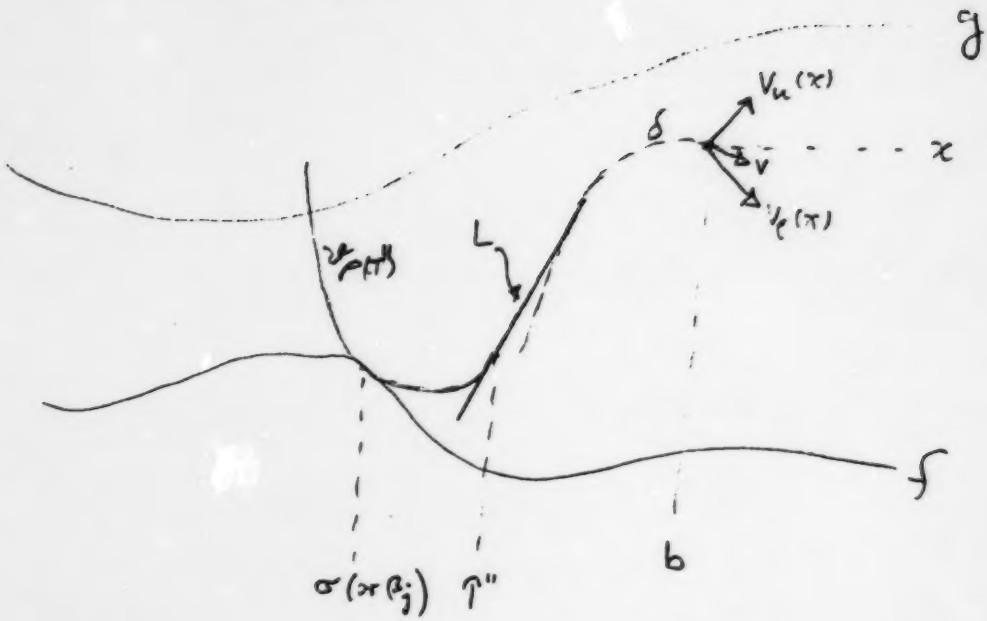

Figure 10: *Proof of Theorem 5*

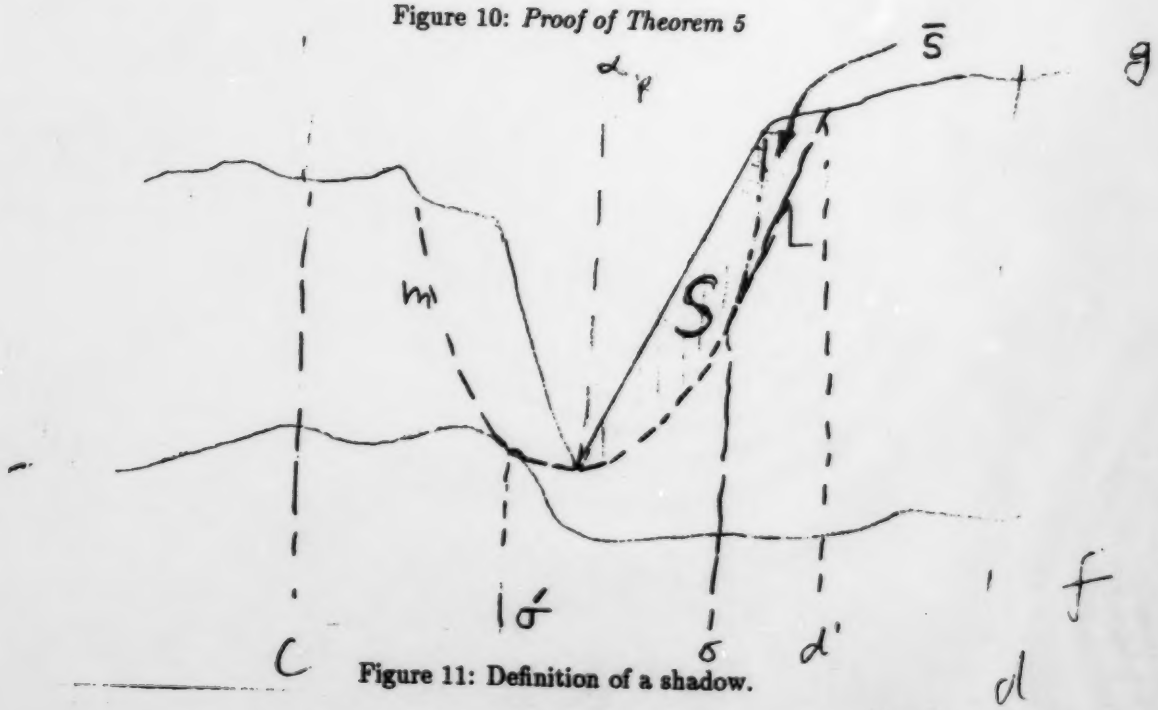

Figure 11: **Definition of a shadow.**

Def (Shading): Consider an interval $[c,d] \subseteq [a,b]$ where there is only one concave vertex in G in the interval $[c,d]$, at $t = \alpha_k \in [c,d]$ (see Figure 11). Consider the parabola m with acceleration M that passes through the point $(\alpha_k, g(\alpha_k))$ and touches F tangentially at some time $\sigma' \leq \alpha_k$ and does not intersect the interior of F anywhere in $[a,\alpha_k]$. Now consider splicing to m the straight line trajectory L with velocity V at time σ. The portion of this spliced trajectory starting at time α_k and ending at the earliest time $d' > \alpha_k$ of intersection of the trajectory with g ($d' = b$, if no such intersection exists on $[\alpha_k, b]$) is called *the shadow of g w.r.t. f* at α_k. The area above the shadow (below g) is called the *shade*. To *shade g w.r.t. f* means to replace g with the shadow on $[\alpha_k, d]$. Also, note that if $m(\sigma) > g(\sigma)$ then L does not exist and the end of the shadow is at the end of the intersection of m and g. Similarly, the definition of the shadow of f *w.r.t.* g is derived by symmetry.

Theorem 6: No point above the shadow of g *w.r.t.* f is reachable.

Proof: (see Figure 11) Consider a point $(\tau, x, v) \in S$. Then $v_{(\tau,x,v)}$ must cross m transversally. Since m touches the boundary of F at some time $\sigma < \alpha_k$ tangentially (or through $f(\beta_j)$ if $\sigma = \beta_j$), therefore by Lemma 3, $v_{(\tau,x,v)}(\sigma) < m(\sigma) \Rightarrow (\tau, x, v)$ is not reachable. Similarly, by symmetry no point below the shadow of f w.r.t. g is reachable.

Now consider a point $p \in \bar{S}$. If p is reachable, then it is on an admissible trajectory entering $\bar{S}$ (not passing through S). But any such trajectory entering $\bar{S}$ must have at some point a velocity greater than V because it crosses L. Hence p is not reachable. ♠

Now we define how to extend the reachable margins of f (g):

Def: Assume g is reachable upto time α_k for some k, and f is reachable upto time β_j, where $(\alpha_k, g(\alpha_k))$ is a concave vertex of G, and $(\beta_j, f(\beta_j))$ is a concave vertex of F, $\beta_j \leq \alpha_k$. To *extend the f-margin across to* α_k means to compute f^* as follows:
(i) find the smallest p such that $p > j$ and $(\beta_p, f(\beta_p))$ is a concave vertex of F.
(ii) Compute the shadow of f w.r.t. g at β_j (if it exists) and fill the result.
(iii) if $\beta_p < \alpha_k$ then $j \leftarrow p$ and goto step (i), else halt.

The definition of g^* follows by symmetry. The motivation behind the algorithm is as follows. Assume that g is reachable upto a given vertex. Then for all the concave vertices of F before the occurrence of the vertex on G (say at time α_k), we can construct the shadows of f *w.r.t.* g and by the reachability of g, these shadows will also be reachable from the initial time. Due to the filling operation, f will be made reachable at least upto time α_k. Similarly the algorithm can extend the margin on g.

Theorem 7: Assume g is reachable upto time α_k and f is reachable upto time β_j, where $(\beta_j, f(\beta_j))$ is a concave vertex of F and $\beta_j \leq \alpha_k$. Assume f^* is produced as defined above. Then,
(i) any trajectory $\bar{X}(t)$ defined over $[a,b]$ is admissible w.r.t. (f,g) *iff* it is admissible w.r.t. (f^*,g).
(ii) if f^* and g do not intersect in $[a,\alpha_k]$, then they form a reachable pair upto time α_k.

Proof:
(i) Since $f^*(t) \geq f(t)\ \forall t \in [a,b]$, any trajectory not admissible w.r.t. (f,g) is not admissible w.r.t. (f^*,g). Conversely, assume $\bar{X}(t)$ is an admissible trajectory w.r.t. (f,g). Since f^* removes only

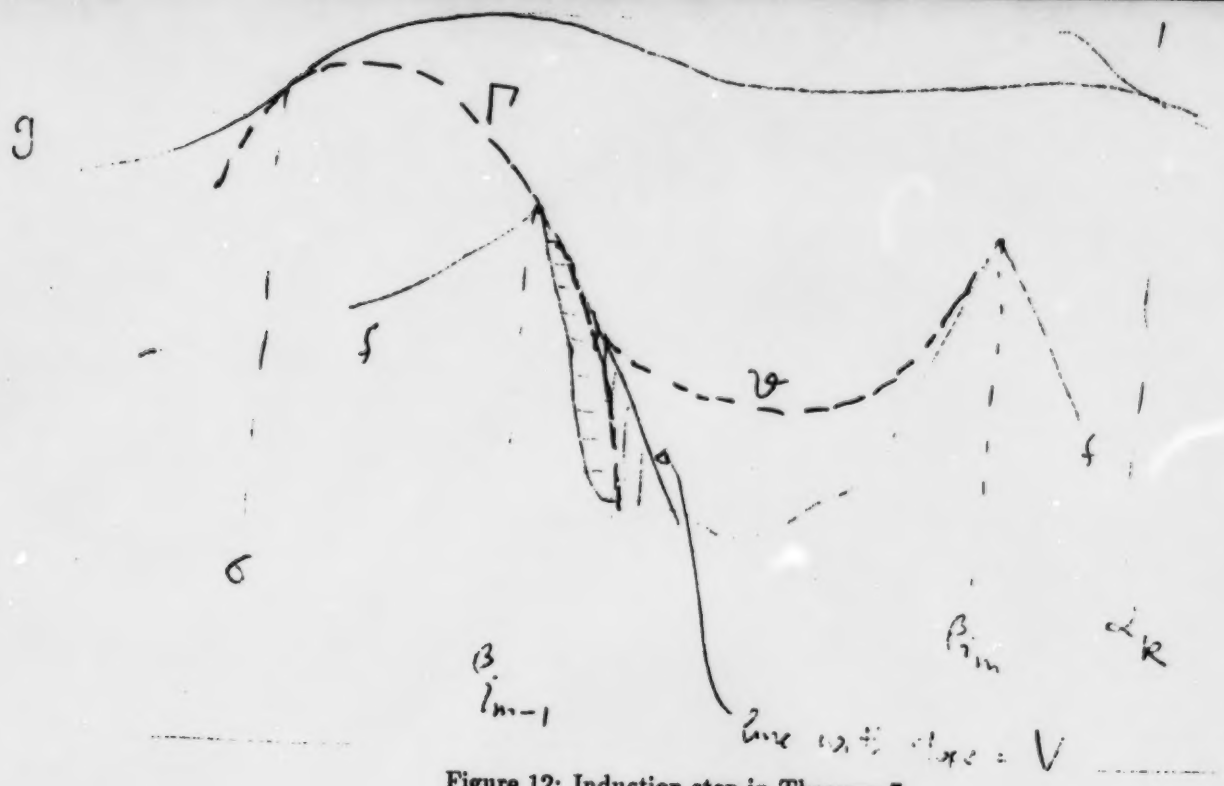

Figure 12: Induction step in Theorem 7

unreachable regions (shading) and regions that are not on any admissible trajectory over $[a, b]$ (filling), hence $\bar{X}(t)$ is admissible w.r.t. (f^*, g).

(ii) Let $\beta_{i_1}, \beta_{i_2}, ..., \beta_{i_m}$ be the times in $[\beta_j, \alpha_k]$ where concave vertices of F occur, i.e., $(\beta_{i_q}, f^*(\beta_{i_q}))$, $q = 1, m$ are all the concave vertices of F in $[\beta_j, \alpha_k]$. First, we show by induction on β_{i_q} that f^* is reachable upto time β_{i_m}.

Induction step: Assume claim is true for β_{i_m-1}. (see Figure 12) We wish to show that $\forall \tau$, $\beta_{i_{m-1}} < \tau < \beta_{i_m}$, $(\tau, f^*(\tau), f^{*\prime}(\tau))$ is reachable.

f^* on $(\beta_{i_{m-1}}, \beta_{i_m})$ consists of the shadow segment at β_j spliced to the filling segment v. There are two possible cases:

Case 1: v touches the shadow at $\beta_{i_{m-1}}$ tangentially. Now we need to show that the points on the shadow are reachable. Note that Γ touches G at a time $\sigma < \beta_{i_{m-1}}$. If Γ touches g tangentially, then we are done, by the reachability of $(\sigma, g(\sigma), g'(\sigma))$. Otherwise Γ passes through $(\sigma = \alpha_l, g(\alpha_l), v)$ for some $l < k$. Note that $v \in [\nu_l(g(\alpha_l)), \nu_u(g(\alpha_l))]$. Hence, we can use Lemma 4 to obtain a trajectory that reaches $(\alpha_l, g(\alpha_l), v)$, and this trajectory can be spliced to Γ to obtain an admissible trajectory that reaches $(\tau, f^*(\tau), f^{*\prime}(\tau))$.

Case 2: v passes through $(\beta_{i_{m-1}}, f(\beta_{i_{m-1}}), v)$, for some v. Since $v \in [\nu_l(f(\beta_{i_{m-1}})), \nu_u(f(\beta_{i_{m-1}}))]$, We can use Lemma 4 to obtain a trajectory that reaches $(\beta_{i_{m-1}}, f(\beta_{i_{m-1}}), v)$. By splicing this trajectory to v, we make $(\tau, f^*(\tau), f^{*\prime}(\tau))$ reachable.

Basis: $(m = 1)$. $\beta_{i_m} = \beta_{i_1} = \beta_j$. Note that on $[a, \beta_j)$, f^* is reachable by assumption.

Now we need to show that f^* is reachable over $[\beta_{i_m}, \alpha_k]$. Since on this interval f^* consists only of a shadow segment (at β_{i_m}) and a filling segment (see Figure 13), we only need to show that the shadow at β_{i_m} is reachable. But this is easily seen by the arguments in (ii) above. Hence f^* is reachable on $[\beta_i, \alpha_k]$. ♠

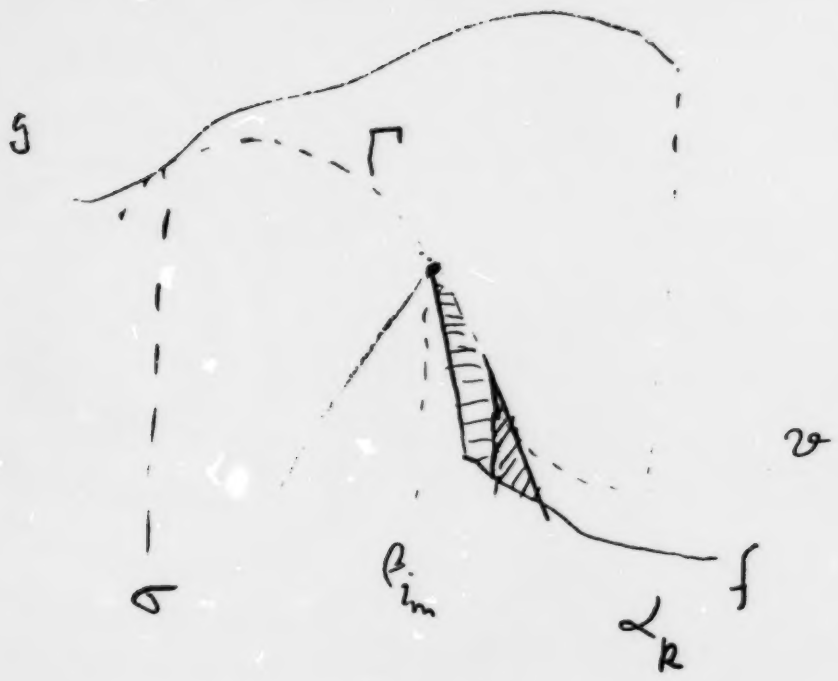

Figure 13: Last case of Theorem 7

5 Algorithmic Results

Our aim is to find out if an admissible trajectory exists satisfying:

$$\bar{X}(a) = x_a,\ \bar{X}'(a) = v_a$$

and

$$\bar{X}(b) = x_b,\ \bar{X}'(b) = v_b$$

This can be done by a pre-processing step that modifies the margins by adding two new parabolic segments with accelerations $+M$ and $-M$ at the point (a, x_a, v_a). Then by repeated application of "extending the margins", two new pursuit functions are created. If these do not intersect transversally anywhere in $[a, b]$, then they form a reachable pair upto time b, with respect to the initial condition. Now, using the method of Lemma 4, the reachability of (b, x_b, v_b) can be easily verified, and repeated applications of Lemma 4 produce the desired trajectory.

Let n and m denote the number of vertices in the upper and lower pursuit functions respectively. Since the algorithm basically consists of repeated applications of "extending the margins", it is fairly straightforward to show that the margins can be made reachable in $O(nm)$. To find the shadow at a given vertex at time β_j (or α_i), in the worst case, we may need to search the entire opposite-chain to find the beginning of the trail of the shadow. Hence for all the vertices, shading takes $O(nm)$ time. The filling operation can be conducted in a manner similar to the Graham scan for convex hulls [Preparata 1985], and hence takes $O(n + m)$ time. Thus, the total run time of the algorithm is $O(nm)$.

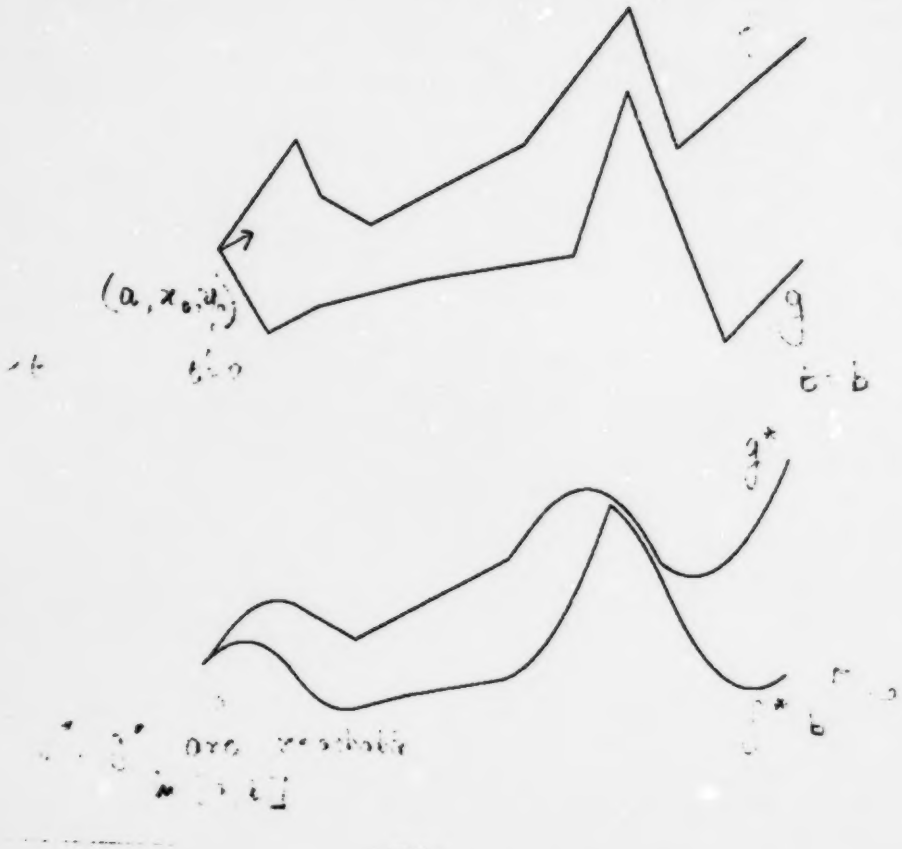

Figure 14: Example 1.

6 Implementation and Examples

We implemented the algorithm and an example is shown in Figures 14 and 15 showing the effect of velocity and acceleration constraints. In Figure 14, the maximum velocity is 3 units, and the maximum acceleration .12 units (same units of time and distance). We also impose one set of boundary conditions, i.e., start position and velocity are specified. The original piecewise linear pursuit functions are shown in Figure 14(a). The reachable pursuit functions after filling and shading are shown in (b). The same example is shown again in Figure 15; however, the acceleration limit has been reduced to 0.05 units. Note that the pruned pursuit functions now intersect, indicating that no admissible collision-free trajectory exists over the whole interval.

7 Conclusion

In this paper, we have given polynomial time ($O(n^2)$) algorithms for 1-D trajectory planning problem in time-varying environments with bounds on robot's velocity and acceleration. The problem essentially involves finding a collision-free trajectory (satisfying the velocity and acceleration bounds) within a simply connected region in space-time. A main aspect of the problem is taking into account the bounds on velocity and acceleration of the robot.

A main motivation for this problem arises in planning the velocity of a robot along a fixed path while avoiding collisions with the obstacles moving across its path. As mentioned in the introduction, this problem can be transformed into planning a collision-free trajectory in a multiply

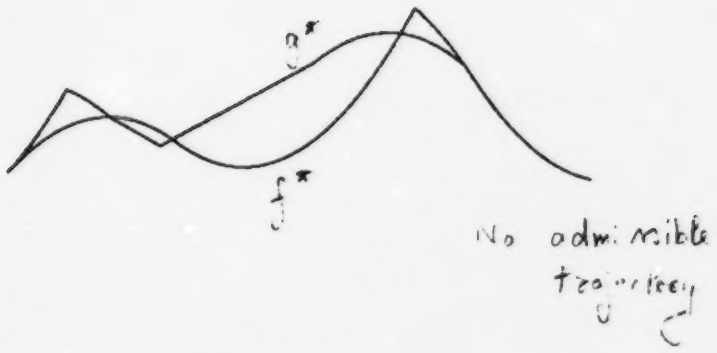

Figure 15: Example 2.

connected space-time region (see Figure 1). This problem was solved in [Kant and Zucker 86,87] with bounds only on the velocity of the robot. We hope to extend the methods used in this paper to solve this more general problem with acceleration bounds as well.

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